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ORIGINAL PAPER

The principal measure and distributional (p, q)-chaos of a coupled lattice system with coupling constant $\varepsilon = 1$ related with Belusov–Zhabotinskii reaction

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Abstract We consider the following system coming from a lattice dynamical system stated by Kaneko (Phys Rev Lett, 65:1391–1394, 1990) which is related to the Belusov–Zhabotinskii reaction:

$$x_n^{m+1} = (1-\varepsilon) f\left(x_n^m\right) + \frac{1}{2}\varepsilon\left[f(x_{n-1}^m) + f\left(x_{n+1}^m\right)\right],$$

where m is discrete time index, n is lattice side index with system size L (i.e., $n = 1, 2, \ldots, L$), $\varepsilon \geq 0$ is coupling constant, and $f(x)$ is the unimodal map on I (i.e., $f(0) = f(1) = 0$, and f has unique critical point c with $0 < c < 1$ and $f(c) = 1$). In this paper, we prove that for coupling constant $\varepsilon = 1$, this CML (Coupled Map Lattice) system is distributionally (p, q)-chaotic for any $p, q \in [0, 1]$ with $p \leq q$, and that its principal measure is not less than $\mu_p(f)$. Consequently, the principal measure of this system is not less than $\frac{2}{3} + \sum_{n=2}^{\infty} \frac{1}{n} \frac{2^{n-1}}{(2^n+1)(2^{n-1}+1)}$ for coupling constant $\varepsilon = 1$ and the tent map Λ defined by $\Lambda(x) = 1 - |1 - 2x|$, $x \in [0, 1]$. So, our results complement the results of Wu and Zhu (J Math Chem, 50:2439–2445, 2012).

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1 Introduction

By a topological dynamical system (t.d.s. for short) (X, f) we mean a compact metric space X together with a continuous selfmap f of X. Since Li and Yorke [1] introduced the term of chaos in 1975, topological dynamical systems were highly discussed and investigated in the literature (see [2,3]) as they are very good examples of problems coming from the theory of topological dynamics and model many phenomena from biology, physics, chemistry, engineering and social sciences.

In physical and chemical engineering applications, such as digital filtering, imaging and spatial vibrations of the elements which compose a given chemical product, one has generalized classical discrete dynamical systems to Lattice Dynamical Systems or 1d Spatiotemporal Discrete Systems which have recently appeared as an interesting and important subject for investigation. In the next section we recall all the definitions. To see the importance of these type of systems, one can refer to [4].

To explore when one of this type of systems has a complicated dynamics or not by the observation of one topological dynamical property is an open and important problem (see [5]). In [5], the authors used the notion of chaos and characterized the dynamical complexity of a coupled lattice system stated by Kaneko in [6] (for more details see for references therein). This CML (Coupled Map Lattice) system is related to the Belusov–Zhabotinskii reaction, and it was proved that this CML (Coupled Map Lattice) system is chaotic in the sense of Li–Yorke and in the sense of Devaney for zero coupling constant (see [5]). Moreover, some problems on the dynamics of this system were given by them for the case of having non-zero coupling constant. Recently, in [7] the authors showed that this system with non-zero coupling constants $\varepsilon \in (0, 1)$ is chaotic in the sense of Li–Yorke and has positive entropy.

The concept of distributional chaos which was introduced by Schweizer and Smítal in [8], is very interesting and important, mainly because it is equivalent to positive topological entropy and some other concepts of chaos when restricted to some spaces (see [8,9]). It is well known that this equivalence does not transfer to higher dimensions, e.g. positive topological entropy does not imply distributional chaos in the case of triangular maps of the unit square [10] (the same may happen when the dimension is zero [11]). To show that there exists a distributional chaotic minimal system, the authors presented an example (see [12]).

More recently, in [13] they showed that for any $p, q \in [0, 1]$ with $p \leq q$, the following coupled lattice system with non-zero coupling constants is distributionally (p, q)-chaotic:

$$x_n^{m+1} = (1-\varepsilon) f(x_n^m) + \frac{1}{2}\varepsilon \left[f(x_{n-1}^m) + f(x_{n+1}^m) \right], \tag{1}$$

where m is discrete time index, n is lattice side index with system size L (i.e., $n = 1, 2, \ldots, L$), $\varepsilon \geq 0$ is coupling constant, and $f(x)$ is the unimodal map on I (i.e., $f(0) = f(1) = 0$ and f has unique critical point c with $0 < c < 1$ and $f(c) = 1$.)

They also proved that the principal measure of this system is not less than $\frac{2}{3} + \sum_{n=2}^{\infty} \frac{1}{n} \frac{2^{n-1}}{(2^n+1)(2^{n-1}+1)}$ for non-zero coupling constants and the tent map Λ defined by $\Lambda(x) = 1 - |1 - 2x|, x \in [0, 1]$. Motivated by the above results, we will continue to study the dynamical properties of the lattice dynamical systems with coupling constant $\varepsilon = 1$. In particular, we prove that for coupling constant $\varepsilon = 1$, this CML (Coupled Map Lattice) system is distributionally (p, q)-chaotic for any $p, q \in [0, 1]$ with $p \le q$, and that its principal measure is not less than $\mu_p(f)$. Consequently, the principal measure of this system is not less than $\frac{2}{3} + \sum_{n=2}^{\infty} \frac{1}{n} \frac{2^{n-1}}{(2^n+1)(2^{n-1}+1)}$ for coupling constant $\varepsilon = 1$ and the tent map Λ. Hence, our results complement the results of Wu and Zhu in [13].

2 Preliminaries

Firstly we complete some notations and recall some concepts. Throughout this paper, $I = [0, 1]$, and X is a compact metric space with metric d.

A pair of points $x, y \in X$ with $x \ne y$ is called a Li–Yorke pair of system (X, f) if the following conditions hold:

(1) $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$.
(2) $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$.

A subset $S \subset X$ is called a LY-scrambled set for f (Li–Yorke set) if the set S has at least two points and every pair of distinct points in S is a Li–Yorke pair. A system (X, f) or a map $f : X \to X$ is said to be chaotic in the sense of Li–Yorke if it has an uncountable scrambled set.

Let (X, f) be a t.d.s.. For any $x, y \in X$ and any $n \in \mathbb{N}$, the distributional function $F_{xy}^n : \mathbb{R}^+ \to [0, 1]$ is defined by

$$F_{xy}^n(t) = \frac{1}{n}\sharp\{i \in \mathbb{N} : d(f^i(x), f^i(y)) < t, 1 \le i \le n\},$$

where $\mathbb{R}^+ = [0, +\infty)$ and $\sharp$ denotes the cardinality. Let

$$F_{xy}(t, f) = \liminf_{n\to\infty} F_{xy}^n(t)$$

and

$$F_{xy}^*(t, f) = \limsup_{n\to\infty} F_{xy}^n(t).$$

For any given $p, q \in [0, 1]$ with $p \le q$, a t.d.s. (X, f) or a continuous map $f : X \to X$ is distributionally (p, q)-chaotic if there exist an uncountable subset $S \subset X$ and $\varepsilon > 0$ such that $F_{xy}(t, f) = p$ and $F_{xy}^*(t, f) = q$ for any $x, y \in S$ with $x \ne y$ and any $t \in (0, \varepsilon)$. Clearly, (X, f) is distributionally chaotic if it is distributionally $(0, 1)$-chaotic (see [13,14]).

The principal measure $\mu_p(f)$ of a t.d.s. (X, f) or a continuous selfmap f of X is defined by

$$\mu_p(f) = \sup_{x,y\in X} \frac{1}{D} \int_0^{+\infty} \left(F^*_{xy}(t, f) - F_{xy}(t, f)\right) dt$$

where $D =$diam (X) is the diameter of the space X (see [15]). It is known that

$$\mu_p(\Lambda) = \frac{2}{3} + \sum_{n=2}^{\infty} \frac{1}{n} \frac{2^{n-1}}{(2^n + 1)(2^{n-1} + 1)},$$

where Λ is the tent map(see [15]).

The state space of LDS (Lattice Dynamical System) is the set

$$\mathcal{X} = \{x : x = \{x_i\}, \quad x_i \in \mathbb{R}^a, i \in \mathbb{Z}^b, \|x_i\| < \infty\}.$$

where $a \geq 1$ is the dimension of the range space of the map of state x_i, $b \geq 1$ is the dimension of the lattice and the l^2 norm

$$\|x\|_2 = \left(\sum_{i\in\mathbb{Z}^b} |x_i|^2\right)^{\frac{1}{2}}$$

is usually taken ($|x_i|$ is the length of the vector x_i) (see [5]).

We will study the following Coupled Map Lattice system posed by Kaneko in [6] (for more details see for references therein) which is related to the Belusov–Zhabotinskii reaction (see [16], and see [17–19] for experimental study of chemical turbulence by this method):

$$x_n^{m+1} = (1 - \varepsilon) f(x_n^m) + \frac{1}{2}\varepsilon \left[f(x_{n-1}^m) + f(x_{n+1}^m)\right], \tag{2}$$

where m is discrete time index, n is lattice side index with system size L, $\varepsilon \geq 0$ is coupling constant, and $f(x)$ is the unimodal map on I.

In general, one of the following periodic boundary conditions of system (1) or (2) is assumed:

1) $x_n^m = x_{n+L}^m$,
2) $x_n^m = x_n^{m+L}$,
3) $x_n^m = x_{n+L}^{m+L}$,

standardly, the first case of the boundary conditions is needed.

3 Main results

The system (2) was investigated by many authors, mostly experimentally or semi-analytically than analytically. The first paper with analytic results is [20], where the authors proved that this system is chaotic in the sense of Li–Yorke. In [5] the authors presented an alternative and easier proof of this result.

Let d be the product metric on the product space I^L, i.e.,

$$d((x_1, x_2, \ldots, x_L), (y_1, y_2, \ldots, y_L)) = \left(\sum_{i=1}^{L} (x_i - y_i)^2 \right)^{\frac{1}{2}}$$

for any $(x_1, x_2, \ldots, x_L), (y_1, y_2, \ldots, y_L) \in I^L$.

Define a map

$$F : (I^L, d) \to (I^L, d)$$

by

$$F(x_1, x_2, \ldots, x_L) = (y_1, y_2, \ldots, y_L)$$

where

$$y_i = (1 - \varepsilon) f(x_i) + \frac{\varepsilon}{2} (f(x_{i-1}) - f(x_{i+1})).$$

It is clear that system (2) is equivalent to the above system (I^L, F) (see [13]). In [5] the authors stated that for non-zero couplings constant, this lattice dynamical system (2) is more complicated.

In [13] the authors proved that the system (2) is distributionally (p, q)-chaotic for any $p, q \in [0, 1]$ with $p \le q$ and any $\varepsilon \in (0, 1)$ when $f = \Lambda$. Motivated by this result and its proof we have the following result.

Theorem 3.1 *The system (2) is distributionally (p, q)-chaotic for any $p, q \in [0, 1]$ with $p \le q$ and $\varepsilon = 1$ when $f = \Lambda$.*

Proof By Proposition 3 in [13], for any $p, q \in [0, 1]$ with $p \le q$, the tent map Λ is distributionally (p, q)-chaotic, i.e., there exist an uncountable subset $A \subset I$ and $\delta > 0$ such that for any $x, y \in A$ with $x \neq y$ and any $0 < t < \delta$,

$$F_{xy}(t, \Lambda) = p \tag{3}$$

and

$$F^{xy}(t, \Lambda) = q. \tag{4}$$

Write

$$\mathcal{F} = \{(x, x, \ldots, x) \in I^L : x \in A\}$$

and

$$\overrightarrow{x} = (x, x, \ldots, x)$$

for any $(x, x, \ldots, x) \in \mathcal{F}$. Then, for any $\overrightarrow{x} = (x, x, \ldots, x)$, $\overrightarrow{y} = (y, y, \ldots, y) \in \mathcal{F}$ with $\overrightarrow{x} \neq \overrightarrow{y}$ and any $n \in \mathbb{N}$, we obtain that

$$F^n(\overrightarrow{x}) = \overrightarrow{\Lambda^n(x)} = (\Lambda^n(x), \Lambda^n(x), \ldots, \Lambda^n(x)) \tag{5}$$

and

$$F^n(\overrightarrow{y}) = \overrightarrow{\Lambda^n(y)} = (\Lambda^n(y), \Lambda^n(y), \ldots, \Lambda^n(y)). \tag{6}$$

By (3), (4), (5) and (6), we have that for any $t \in (0, \sqrt{L}\delta)$,

$$\begin{aligned} F_{\overrightarrow{x}\overrightarrow{y}}(t, F) &= \liminf_{n\to\infty} \frac{1}{n} \sharp\{i \in \mathbb{N} : d(F^i(\overrightarrow{x}), F^i(\overrightarrow{y})) < t, 1 \leq i \leq n\} \\ &= \liminf_{n\to\infty} \frac{1}{n} \sharp\{i \in \mathbb{N} : |\Lambda^i(x), \Lambda^i(y)| < \frac{t}{\sqrt{L}}, 1 \leq i \leq n\} \\ &= F_{xy}(\frac{t}{\sqrt{L}}, \Lambda) \end{aligned} \tag{7}$$

and

$$\begin{aligned} F^{\overrightarrow{x}\overrightarrow{y}}(t, F) &= \limsup_{n\to\infty} \frac{1}{n} \sharp\{i \in \mathbb{N} : d(F^i(\overrightarrow{x}), F^i(\overrightarrow{y})) < t, 1 \leq i \leq n\} \\ &= \limsup_{n\to\infty} \frac{1}{n} \sharp\{i \in \mathbb{N} : |\Lambda^i(x), \Lambda^i(y)| < \frac{t}{\sqrt{L}}, 1 \leq i \leq n\} \\ &= F^{xy}(\frac{t}{\sqrt{L}}, \Lambda). \end{aligned} \tag{8}$$

As A is uncountable, $\mathcal{F}$ is uncountable. Consequently, by the definition the system (2) is distributionally (p, q)-chaotic. Thus, the proof is completed.

In [13] the authors proved that the principal measure of the system (1) is not less than $\mu_p(f)$ for any $\varepsilon \in (0, 1)$. Motivated by this result and its proof we abtain the following result.

Theorem 3.2 *The principal measure of the system (2) is not less than $\mu_p(f)$ for $\varepsilon = 1$.*

Proof For any $\overrightarrow{x} = (x, x, \ldots, x)$, $\overrightarrow{y} = (y, y, \ldots, y) \in I^L$ with $\overrightarrow{x} \neq \overrightarrow{y}$ and any $n \in \mathbb{N}$, it is easily seen that for the system (2),

$$F^n(\overrightarrow{x}) = (f^n(x), f^n(x), \ldots, f^n(x))$$

and

$$F^n(\overrightarrow{y}) = (f^n(y), f^n(y), \ldots, f^n(y)).$$

This means that

$$F_{\overrightarrow{x}\,\overrightarrow{y}}(\sqrt{L}t, F) = F_{xy}(t, f)$$

and

$$F^{\overrightarrow{x}\,\overrightarrow{y}}(\sqrt{L}t, F) = F^{xy}(t, f)$$

for any $\overrightarrow{x}, \overrightarrow{y} \in I^L$. Write $D = diam(I^L)$. As $|I| = 1$,

$$\begin{aligned}
\mu_p(F) &\geq \sup_{x,y\in I} \frac{1}{D} \int_0^{+\infty} \left(F^*_{\overrightarrow{x}\,\overrightarrow{y}}(t, F) - F_{\overrightarrow{x}\,\overrightarrow{y}}(t, F)\right) dt \\
&= \sup_{x,y\in I} \frac{1}{\sqrt{L}} \int_0^{+\infty} \left(F^*_{\overrightarrow{x}\,\overrightarrow{y}}(t, F) - F_{\overrightarrow{x}\,\overrightarrow{y}}(t, F)\right) dt \\
&= \sup_{x,y\in I} \frac{1}{\sqrt{L}} \int_0^{+\infty} \left(F^*_{xy}\left(\frac{t}{\sqrt{L}}, f\right) - F_{xy}\left(\frac{t}{\sqrt{L}}, f\right)\right) dt \\
&= \sup_{x,y\in I} \int_0^{+\infty} \left(F^*_{xy}(t, f) - F_{xy}(t, f)\right) dt = \mu_p(f).
\end{aligned}$$

Thus, the proof is completed.

Remark 3.1 From [13] one can see that the principal measure of system (1) or (2) is not less than $\frac{2}{3} + \sum_{n=2}^{\infty} \frac{1}{n} \frac{2^{n-1}}{(2^n+1)(2^{n-1}+1)}$ for any non-zero coupling constant $\varepsilon \in (0, 1)$ and the tent map Λ. However, by Eq. (1) in [13] and Theorem 3.2, the principal measure of system (1) or (2) is not less than $\frac{2}{3} + \sum_{n=2}^{\infty} \frac{1}{n} \frac{2^{n-1}}{(2^n+1)(2^{n-1}+1)}$ for coupling constant $\varepsilon = 1$ and the tent map Λ. Therefore, our results complement the results of Wu and Zhu in [13].

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